

Static Optimization with MATLAB

Day 2: Constrained Optimization with Equality Constraints

GSEFM Pre-Semester Course

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Problem Setting I

Definition (Optimization Problem with Equality Constraints)

An optimization problem with equality constraints consists of a set of feasible solutions M , given as:

$$M = \{x \in \mathbb{R}^n \mid h(x) = 0\},$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and an objective function $f : M \rightarrow \mathbb{R}$. The goal is to find the maximum value of $f(x)$ over all $x \in M$, which is expressed as:

$$\max_{x \in M} f(x)$$

Problem Setting II

- We assume that f and the equality constraints $h_1, \dots, h_m : \mathbb{R}^n \rightarrow \mathbb{R}$ are twice continuously differentiable.
- Usually, the equality constraints are given by

$$h_i(x) - c_i = 0 \text{ for all } i = 1, \dots, m$$

- We need that $m \leq n$, i.e., to have fewer constraints than choice variables.

The Lagrangian I

Definition (Lagrangian Function)

The Lagrangian function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is defined as

$$\mathcal{L}(x, \lambda) := f(x) - \sum_{i=1}^m \lambda_i [h_i(x) - c_i],$$

where the components $\lambda_1, \dots, \lambda_m$ of λ are called Lagrange multipliers (often also the vector λ is called Lagrange multiplier).

Then we try to solve the problem

$$\max_{x, \lambda} \mathcal{L}(x, \lambda) = \max_{x, \lambda_1, \dots, \lambda_m} f(x) - \sum_{i=1}^m \lambda_i [h_i(x) - c_i]$$

The Lagrangian II

- Idea: turn a constrained problem into an unconstrained problem
- Economic Intuition: "allow" the agent to violate the constraint at a constant "tax rate" (shadow price) λ (to be determined) per unit violation, such that in optimum, it will not be violated
- The agent's payoff, net of the "penalty" is then given by the Lagrangian function.

Constrained Optimization: Solution Strategy

- Let $x^*(\lambda)$ be the solution to $\max_x \mathcal{L}(x, \lambda)$, found by the FOCs

$$\frac{\partial \mathcal{L}(x, \lambda)}{\partial x}(x^*) = \mathbf{0}$$

⇒ But, given an arbitrary λ , there is no guarantee that the solution is the optimal solution to the original problem

- Thus, find the right λ^* such that $[h_i(x^*(\lambda^*)) - c_i] = 0$ for all i , i.e.,

$$\frac{\partial \mathcal{L}(x, \lambda)}{\partial \lambda}(x^*) = \mathbf{0}$$

- Then, we have a system of $n + m$ equations which we can solve for x^* and λ^* .

Lagrange Theorem I

Theorem (Lagrange Theorem)

Let x^* be an optimal solution of

$$\max_x f(x) \quad \text{s.t.} \quad h(x) = \mathbf{0}$$

where f and h are continuously differentiable functions and h fulfills the so-called **non-degenerate constraint qualification (NDCQ)**, i.e., $\text{rank}(Dh(x^*)) = m < n$

► *Definition*. Then there exist unique Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ such that

$$Df(x^*) - \lambda^{*T} Dh(x^*) = \mathbf{0}$$

with λ being a column vector of Lagrange multipliers.

Lagrange Theorem II

- ⇒ The last equation states that the gradient of the objective function can be written as a linear combination of the gradients of the constraint functions (evaluated at the optimal solution).
- ⇒ this is of course a necessary condition.

Lagrange Theorem: Intuition I

⇒ It is not really clear why optimizing the Lagrangian, an artificial unconstrained problem, in (x, λ) should give us the solution to the original constrained problem, i.e.,

$$\max_x f(x) \text{ s.t. } h(x) = \mathbf{0}.$$

⇒ The crucial question is why the Lagrangian condition,

$$Df(x^*) - \lambda^{*T} Dh(x^*) = \mathbf{0},$$

which is simply the **FOC of the Lagrangian**, holds in an optimal point x^* ?

Lagrange Theorem: Intuition II

Consider the simple problem with 2 decision variables and 1 constraint:

$$\max_{x_1, x_2} f(x_1, x_2) \quad s.t. \quad g(x_1, x_2) = c$$

The Lagrangian gives

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda[g(x_1, x_2) - c]$$

Lagrange Theorem: Intuition III

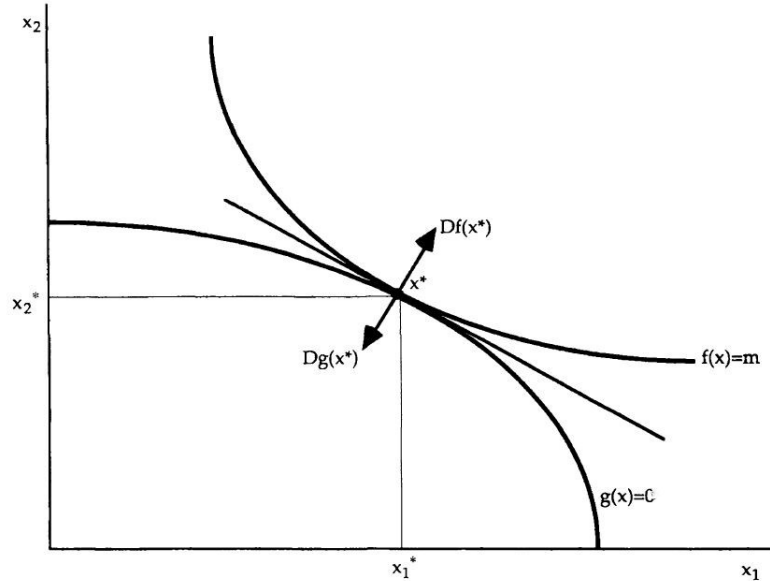


Figure 7.3. Optimal solution of a Lagrange problem.

Lagrange Theorem: Intuition IV

- The feasible set of the problem ($g(x) = c$) is a curve on the (x_1, x_2) -plane
- The optimal solution to the problem is the point $x^* = (x_1^*, x_2^*)$ on this curve that lies on the highest possible level set of f , $f(x) = m$ in the picture.

Lagrange Theorem: Intuition V

- Given certain convexity assumptions, x^* will be a tangency point of the two level curves
 - This implies that the gradients of f and g (both orthogonal to the tangent) lie on the same straight line
- ⇒ we can write one of them as a product of the other and a scalar: there exists a number λ^* such that

$$Df(x^*) = \lambda^* Dg(x^*)$$

which is another way to write the FOCs of $\mathcal{L}$ w.r.t x :

$$\begin{aligned}\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} &= \frac{\partial f}{\partial x_1}(x_1^*, x_2^*) - \lambda^* \frac{\partial g}{\partial x_1}(x_1^*, x_2^*) = 0 \\ \frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} &= \frac{\partial f}{\partial x_2}(x_1^*, x_2^*) - \lambda^* \frac{\partial g}{\partial x_2}(x_1^*, x_2^*) = 0\end{aligned}$$

Non-Degenerate Constraint Qualification I

Definition (Non-Degenerate Constraint Qualification)

For n variables and m constraints, the Jacobian of the constraint vector $h(x) = [h_1(x), \dots, h_m(x)]'$ is

$$Dh(x) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(x) & \cdots & \frac{\partial h_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(x) & \cdots & \frac{\partial h_m}{\partial x_n}(x) \end{pmatrix}$$

We say that $(h_1, \dots, h_m)$ satisfy the NDCG at x if the rank of the Jacobian matrix at that point, $Dh(x)$, is m , that is $Dh(x)$ has full rank.

This condition is usually also called **LICQ** (Linear Independent Constraint Qualification).

Non-Degenerate Constraint Qualification II

The condition can be violated in two ways:

- if one row has all its elements equal to zero
- if one row can be represented by a linear combination of the others

Importance of the Constraint Qualification

Note that the Lagrange condition can be rewritten as

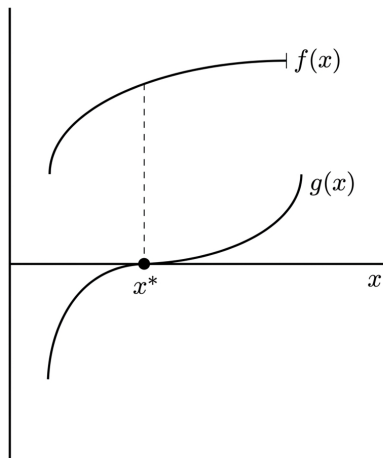
$$Df(x^*) = \lambda^T Dh(x^*) = \sum_{j=1}^m \lambda_j Dh^j(x^*)$$

- ⇒ requires that we can write the gradient of f as a linear combination of the gradients of the constraint functions.
- ⇒ If the constraint qualification fails, we may not have enough linearly independent constraint gradients to do so.

Failure of Constraint Qualification

Observe two cases where x^* maximizes f under the constraint $g(x) = \mathbf{0}$ (because it is the only point in the feasible set). But the Lagrange condition is not satisfied:

Failure of Constraint Qualification: Univariate Case

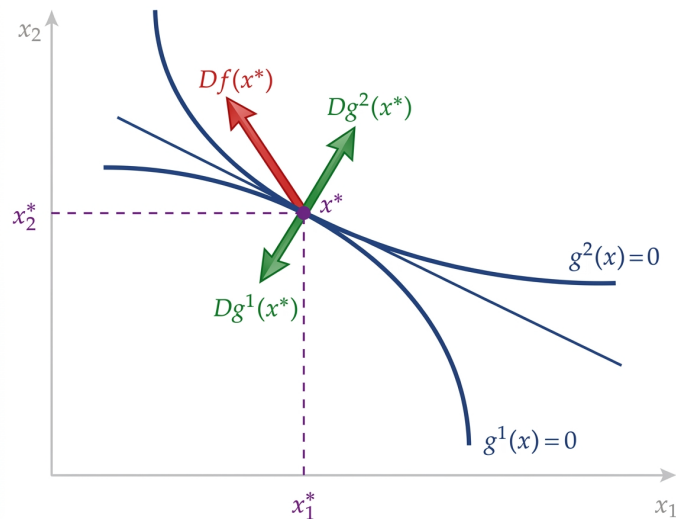


In this case, the point which maximizes $f(x)$ is also a critical point of $g(x)$, i.e. $Dg(x^*) = 0$. As the necessary condition for a maximum is

$$Df(x^*) = \lambda^* Dg(x^*)$$

this would only hold if $Df(x^*) = 0$. However, since $Df(x^*) > 0$, there is no number λ such that the necessary condition holds.

Failure of Constraint Qualification: Multivariate Case



The zero level curves of the two constraints are tangent at the optimum $\Rightarrow$ the gradients of the constraint functions lie on the same straight line and therefore do not span the (x_1, x_2) -plane $\Rightarrow Df(x^*)$ cannot be written as a linear combination of $Dg^1(x^*)$ and $Dg^2(x^*)$ unless, by chance, it lies on the same straight line.

Checking the Constraint Qualification: Exercise 1

- Consider the two constraints:

$$x^2 + y^2 = 1, \text{ and } x + z = 1.$$

- Can the constraint qualification fail for any point $(x, y, z) \in \mathbb{R}^3$?

Checking the Constraint Qualification: Solution 1

The Jacobian gives:

$$Dh(x, y, z) = \begin{bmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It becomes problematic if $x = y = 0$: then the rank of $Dh(x, y, z)$ is only equal to 1. But this violates the constraint h_1 and thus is not a point in the feasible set. Here one does not have to worry about the NDCQ.

The non-degenerate constraint qualification: Exercise 2

Consider the problem

$$\begin{aligned} \max f(x, y) &= x \\ \text{s.t. } x^3 + y^2 &= 0. \end{aligned}$$

Solve the constrained maximization problem and check for the non-degenerate constraint qualification.

The non-degenerate constraint qualification: Solution 2 I

Write the Lagrangian

$$L = x - \lambda(x^3 + y^2),$$

and take the FOCs

$$1 - 3\lambda x^2 = 0$$

$$-2\lambda y = 0$$

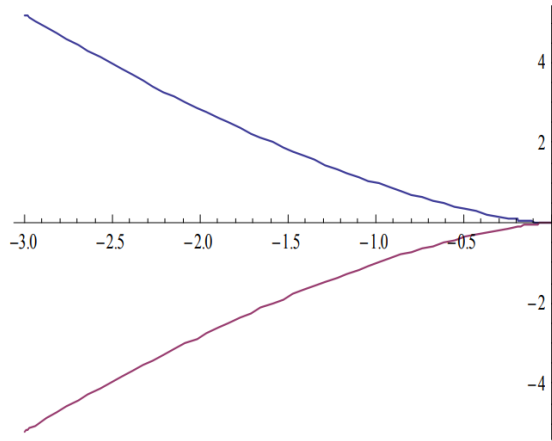
$$x^3 + y^2 = 0.$$

The first equation implies that $x \neq 0$ (why?). Using now the third equation we obtain that $y = \pm\sqrt{-x^3} \neq 0$. Since $y \neq 0$ the second equation implies that $\lambda = 0$. But if we plug in $\lambda = 0$ we obtain a contradiction. Therefore the system of the Lagrange equations does not have a solution.

The non-degenerate constraint qualification: Solution 2 II

What is wrong?

The set $(x, y) \in \mathbb{R}^2 : x^3 + y^2 = 0$ is represented in the following figure:



Clearly, the solution of our problem is $x = 0$.

The non-degenerate constraint qualification: Solution 2 III

The Jacobian of h is

$$Dh(x, y) = [3x^2, 2y]$$

The point $(0,0)$ satisfies the restriction of the problem, but

$$\text{rank}(Dh(0,0)) = \text{rank}([0,0]) = 0 \neq 1$$

so that the non-degenerate constraint qualification does not hold.

Sufficient Conditions for a Strict Local Maximum

Theorem (Sufficient condition for a strict local maximum)

Let x^ be a feasible point satisfying the Lagrange condition for some λ^* . Assume that the Hessian of the Lagrangian function with respect to the choice variables x , evaluated at (x^*, λ^*) , given by*

$$D_x^2 \mathcal{L}(x^*, \lambda^*) = D^2 f(x^*) - \lambda^{*'} D^2 h(x^*),$$

is negative definite subject to the constraints $Dh(x^)y = \mathbf{0}$, i.e., it holds that*

$$y' D_x^2 \mathcal{L}(x^*, \lambda^*) y < 0, \text{ for all } y \in \mathbb{R}^n, y \neq 0 \text{ with } Dh(x^*)y = \mathbf{0}.$$

Then x^ is a strict local maximizer of f subject to $h(x) = \mathbf{0}$.*

How to check the additional condition?

- In addition to determining x^* , λ^* , and checking the NDCQ, we need to check the additional condition,

$$y' D_x^2 \mathcal{L}(x^*, \lambda^*) y < 0, \text{ for all } y \in \mathbb{R}^n \text{ with } Dh(x^*)y = \mathbf{0},$$

to conclude that x^* is indeed a strict local maximizer.

- ⇒ The criterion is not easy to check, since one must first find all possible $y \in \mathbb{R}^n$ such that $Dh(x^*)y = \mathbf{0}$, and then check the **negative definiteness of $D_x^2 \mathcal{L}(x^*, \lambda^*)$ on this subset**.
- ⇒ **Way out:** Look at the so-called Bordered Hessian.

Bordered Hessian I

Border the $n \times n$ Hessian of $D_{\mathbf{x}}^2 \mathcal{L}(\mathbf{x}^*, \lambda^*)$ with the $m \times n$ Jacobian matrix of the constraints $D\mathbf{h}(\mathbf{x}^*)$:

$$H \equiv \begin{pmatrix} \mathbf{0} & D\mathbf{h}(\mathbf{x}^*) \\ D\mathbf{h}(\mathbf{x}^*)^T & D_{\mathbf{x}}^2 \mathcal{L}(\mathbf{x}^*, \boldsymbol{\mu}^*) \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{pmatrix}$$

$\Rightarrow$ In the unconstrained case of $m = 0$ the above condition coincides with the condition for the (un)bordered Hessian to be negative definite (or positive definite for a minimum).

Bordered Hessian II

Lemma (Bordered Hessian and the additional condition)

- If the last $(n - m)$ **leading principal minors** of the Bordered Hessian H evaluated at the potential strict local **maximum** alternate in sign, and the smallest one having the sign $(-1)^{m+1}$, then it holds that

$$y' D_x^2 \mathcal{L}(x^*, \lambda^*) y < 0, \text{ for all } y \in \mathbb{R}^n, y \neq 0 \text{ with } Dh(x^*)y = \mathbf{0}.$$

- If the last $(n - m)$ **leading principal minors** of the Bordered Hessian H evaluated at the potential strict local **minimum** have the same sign as $(-1)^m$, then it holds that

$$y' D_x^2 \mathcal{L}(x^*, \lambda^*) y > 0, \text{ for all } y \in \mathbb{R}^n, y \neq 0 \text{ with } Dh(x^*)y = \mathbf{0}.$$

Bordered Hessian III

Consider the following example:

$$\max_{x \in \mathbb{R}^4} f(x) \text{ s.t. } h(x) = \mathbf{0},$$

where $h : \mathbb{R}^4 \rightarrow \mathbb{R}^2$. Then we have that

- $n = 4$ and $m = 2$
- check last $n - m = 4 - 2 = 2$ leading principal minors of the bordered Hessian, namely H_3 and H_4 . The sign condition is applied on the **smallest** matrix (in terms of dimension), namely H_3 .

Exercise 1

Solve the problem

$$\max_{x,y,z} 2x - 2y + z \quad \text{s.t.} \quad x^2 + y^2 + z^2 = 9$$

by the Lagrange method.

Use the sufficient second-order conditions for a strict maximum to determine which of the solutions to the system of first-order conditions yields a maximum.

Exercise 1 Solution I

The Lagrangian is

$$\mathcal{L}(x, y, z, \lambda) = 2x - 2y + z - \lambda[x^2 + y^2 + z^2 - 9]$$

First-order necessary conditions:

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, z^*, \lambda^*) = 2 - 2x^*\lambda^* = 0 \quad \Rightarrow \quad x^* = \frac{1}{\lambda^*}$$

$$\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, z^*, \lambda^*) = -2 - 2y^*\lambda^* = 0 \quad \Rightarrow \quad y^* = -\frac{1}{\lambda^*}$$

$$\frac{\partial \mathcal{L}}{\partial z}(x^*, y^*, z^*, \lambda^*) = 1 - 2z^*\lambda^* = 0 \quad \Rightarrow \quad z^* = \frac{1}{2\lambda^*}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x^*, y^*, z^*, \lambda^*) = x^{*2} + y^{*2} + z^{*2} - 9 = 0$$

Exercise 1 Solution II

Check NDCQ: $Dh(x, y, z) = [2x \ 2y \ 2z] \Rightarrow$ only problematic for $(x^*, y^*, z^*) = (0, 0, 0)$

Exercise 1 Solution III

Substituting the first three into the last equation:

$$\frac{1}{\lambda^{*2}} + \frac{1}{\lambda^{*2}} + \frac{1}{4\lambda^2} = \frac{9}{4\lambda^{*2}} = 9 \Rightarrow \lambda^* = \pm \frac{1}{2}$$

2 candidate solutions:

$$\lambda^* = \frac{1}{2} \Rightarrow (x^*, y^*, z^z) = (2, -2, 1) \Rightarrow f(x^*, y^*, z^z) = 9$$

$$\lambda^* = -\frac{1}{2} \Rightarrow (x^*, y^*, z^z) = (-2, 2, -1) \Rightarrow f(x^*, y^*, z^z) = -9$$

Therefore, the first point should be the maximizer we seek.

To check the sufficient conditions for a strict local maximum, we have to look at the leading principle minors of the bordered Hessian.

Exercise 1 Solution IV

$$H = \begin{bmatrix} 0 & h_x & h_y & h_z \\ h_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} & \mathcal{L}_{xz} \\ h_y & \mathcal{L}_{yz} & \mathcal{L}_{yy} & \mathcal{L}_{yz} \\ h_z & \mathcal{L}_{zx} & \mathcal{L}_{zy} & \mathcal{L}_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{bmatrix}$$

H has to be negative definite subject to the constraints $Dh(x^*) = 0$

$\Rightarrow$ The last two principal minors of H have to alternate in sign and the smallest one having the sign $(-1)^{m+1}$

Exercise 1 Solution V

$k = 3$:

$$\begin{aligned} |H_3| &= \begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{vmatrix} \\ &= -2x(-4x\lambda) + 2y(4y\lambda) \\ &= 8x^2\lambda + 8y^2\lambda \\ &= 8\lambda(x^2 + y^2) > 0. \end{aligned}$$

Exercise 1 Solution VI

$k = 4$:

$$\begin{aligned} |H_4| &= -2x \begin{vmatrix} 2x & 0 & 0 \\ 2y & -2\lambda & 0 \\ 2z & 0 & -2\lambda \end{vmatrix} + 2y \begin{vmatrix} 2x & -2\lambda & 0 \\ 2y & 0 & 0 \\ 2z & 0 & -2\lambda \end{vmatrix} \\ &\quad - 2z \begin{vmatrix} 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \\ 2z & 0 & 0 \end{vmatrix} \\ &= -2x(8x\lambda^2) + 2y(-8y\lambda^2) - 2z(8z\lambda^2) \\ &= -16x^2\lambda^2 - 16y^2\lambda^2 - 16z^2\lambda^2 \\ &= -16\lambda^2(x^2 + y^2 + z^2) < 0. \end{aligned}$$

Exercise 2 Utility Maximization

Consider the problem of a consumer maximizing his utility

$$\max_{x_1, x_2} (\log(x_1) + \log(x_2))$$

subject to the budget constraint

$$3x_1 + 2x_2 = 2$$

Exercise 2 Solution I

Check NDCQ:

$$Dg(x) = \begin{bmatrix} \frac{\partial(3x_1+2x_2)}{\partial x_1} & \frac{\partial(3x_1+2x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3 & 2 \end{bmatrix}$$

satisfied.

Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = \log(x_1) + \log(x_2) - \lambda(3x_1 + 2x_2 - 2)$$

First-order conditions (FOCs):

$$\frac{\partial \mathcal{L}}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = \frac{1}{x_1^*} - 3\lambda^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = \frac{1}{x_2^*} - 2\lambda^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x_1^*, x_2^*, \lambda^*) = 3x_1^* + 2x_2^* - 2 = 0$$

Exercise 2 Solution II

Solve the system of 3 equations with 3 unknowns:

Solve first two equations for x_1^* and x_2^* , respectively:

$$x_1^* = \frac{1}{3\lambda^*}, \quad x_2^* = \frac{1}{2\lambda^*}$$

Plug into the third:

$$3\frac{1}{3\lambda^*} + 2\frac{1}{2\lambda^*} = 2 \quad \Rightarrow \quad \lambda^* = 1$$
$$\Rightarrow x_1^* = \frac{1}{3}, \quad x_2^* = \frac{1}{2}.$$

Exercise 2 Solution III

Lastly, check the sufficient condition:

Bordered Hessian:

$$H = \begin{bmatrix} 0 & 3 & 2 \\ 3 & -\frac{1}{x_1^{*2}} & 0 \\ 2 & 0 & -\frac{1}{x_2^{*2}} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ 3 & -9 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

$$\bullet \quad |H_3| = \begin{vmatrix} 0 & 3 & 2 \\ 3 & -9 & 0 \\ 2 & 0 & -4 \end{vmatrix} = -3(-12) + 2(18) = 36 + 36 = 72 > 0$$

$\Rightarrow$ negative definiteness fulfilled $\Rightarrow$ maximum

Exercise 3 Labour Supply Model

An agent has a utility function $U(c, l)$ where c is consumption and l is leisure. She receives non-labor income G and can additionally decide to work for an hourly wage of w . In total, she divides her overall daily time $T = 24$ into h hours of work and l hours of leisure.

1. Write down the maximization problem and solve for the optimality conditions
2. Calculate the optimal leisure and income assuming that the utility function is given by $U(c, l) = c^\alpha l^{1-\alpha}$ where $\alpha \in (0, 1)$ (Cobb-Douglas).

Exercise 3 Solution I

1. Maximization problem and FOCs:

$$\max_{c,l} U(c, l) \quad \text{s.t.} \quad h + l = 24, \quad G + wh = c$$

Substituting for h , the combined constraint is

$$G + w(24 - l) - c = 0$$

With

$$Dg(c, l) = \begin{bmatrix} -1 & -w \end{bmatrix}$$

the NDCQ is satisfied.

The Lagrangian gives:

$$\mathcal{L}(c, l, \lambda) = U(c, l) - \lambda[G + w(24 - l) - c]$$

Exercise 3 Solution II

The FOCs give:

$$\frac{\partial \mathcal{L}}{\partial c}(c^*, l^*, \lambda^*) = U_c(c^*, l^*) + \lambda^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial l}(c^*, l^*, \lambda^*) = U_l(c^*, l^*) + w\lambda^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}(c^*, l^*, \lambda^*) = G + w(24 - l^*) - c^* = 0$$

2. Now assume the utility function takes the explicit form $U(c, l) = c^\alpha l^{1-\alpha}$:
The first two FOCs become:

$$\begin{aligned}\alpha c^{*\alpha-1} l^{*1-\alpha} &= -\lambda^* \\ (1-\alpha) c^{*\alpha} l^{*- \alpha} &= -w\lambda^*\end{aligned}$$

Exercise 3 Solution III

Dividing them by each other and rearranging gives

$$l^* = \frac{1 - \alpha}{w\alpha} c^*$$

Plugging this into the constraint, one can solve for c^* and then l^* :

$$c^* = \alpha(G + 24w) \Rightarrow l^* = \frac{1 - \alpha}{w}(G + 24w)$$

$\Rightarrow$ goods and leisure consumption is a fixed share of total maximal income.

Lastly, check the bordered Hessian for sufficiency:

$$H(c^*, l^*) = \begin{bmatrix} 0 & -1 & -w \\ -1 & \alpha(\alpha - 1)c^{\alpha-2}l^{1-\alpha} & \alpha(1 - \alpha)c^{\alpha-1}l^{-\alpha} \\ -w & (1 - \alpha)\alpha c^{\alpha-1}l^{-\alpha} & (1 - \alpha)(-\alpha)c^{\alpha}l^{-\alpha-1} \end{bmatrix}$$

Exercise 3 Solution IV

▪ $|H_3| =$

$$\begin{aligned} & 1[-(1-\alpha)(-\alpha)c^{*\alpha}l^{*-\alpha-1} - (-w)\alpha(1-\alpha)c^{*\alpha-1}l^{*-\alpha}] \\ & + (-w)[-1(1-\alpha)\alpha c^{*\alpha-1}l^{*-\alpha} - (-w)\alpha(\alpha-1)c^{*\alpha-2}l^{*1-\alpha}] \\ = & \alpha(1-\alpha)U^* \left[\frac{1}{l^{*2}} + 2w \frac{1}{c^*l^*} + w^2 \frac{1}{c^{*2}} \right] \\ = & \alpha(1-\alpha)U^* \left(\frac{1}{l^*} + \frac{w}{c^*} \right)^2 > 0 \end{aligned}$$

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Rank of a Matrix I

Definition (Rank of a Matrix)

The rank of a matrix A is defined as

- the maximum number of linearly independent column vectors in the matrix A .
 - the maximum number of linearly independent row vectors in the matrix A .
-
- Both definitions are equivalent.
 - When the rank of a matrix A equals the smallest dimension it is called **full rank**.
 - A square matrix is full rank if and only if its determinant is nonzero.

Rank of a Matrix II

Exercises: Are A and B full rank? If not, what is their rank?

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{pmatrix}$$

► Go Back